The critical properties of two-dimensional oscillator arrays

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Abstract
We present a renormalization group study of two-dimensional oscillator arrays,
with dissipative, short-range interactions. We consider the case of non-
identical oscillators, with distributed intrinsic frequencies within the array
and study the steady-state properties of the system. In two dimensions no
macroscopic mutual entrainment is found but, for identical oscillators, critical
behaviour of the Berezinskii–Kosterlitz–Thouless (BKT) type is shown to be
present. We then discuss the stability of BKT order in the physical case of
distributed quenched random frequencies. In order to do that, we show how
the steady-state dynamical properties of the two-dimensional array of non-
identical oscillators are related to the equilibrium properties of the XY model
with quenched randomness, which has already been studied in the past. We
propose a novel set of recursion relations to study this system within the Migdal–
Kadanoff (MK) renormalization group scheme, by means of a discrete clock
state formulation. For identical oscillators, we compute the phase diagram in
the presence of random dissipative coupling, at finite values of the clock state
parameter. Possible experimental applications in two-dimensional arrays of
microelectromechanical oscillators are briefly suggested.

*This work is devoted to Professor David Sherrington, in occasion of his 65th
birthday.*

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1. Introduction

The dynamical properties of large arrays of self-sustained oscillators with distributed intrinsic
frequencies represent an interesting problem, bridging non-equilibrium statistical physics and
nonlinear sciences. Large populations of interacting, self-sustained oscillators are known to
model a great variety of biological systems and progress in their study has been achieved in
recent decades [1–3]. Dense arrays of coupled, self-sustained oscillators, close to an oscillatory

instability of the Hopf type, are expected to present critical behaviour. The analysis of such systems however, in the low-dimensional case, i.e. when the range of the interaction between oscillators decays relatively fast with distance, is a challenging one and very little attempts have been made in this direction.

The mean-field Kuramoto model [1] represents a very special case, where one can predict in a clear way the macroscopic mutual entrainment (MME) properties of the system, but attempts to generalize it to the low-dimensional case are difficult and the majority of the studies on oscillator arrays have been devoted to the mean-field case, and to networks with a relatively high degree of connectivity. On the other hand, the case of short-range interactions is a very interesting one and has potential applications, in particular when considering dissipatively and/or reactively coupled arrays of mechanical oscillators at micro- and nano-scales.

The oscillator lattice problem, namely the finite-dimensional, nearest-neighbour version of the Kuramoto model, has been attacked in the past by several authors [4, 5]. The main conclusions are that the synchronization properties one normally finds in the mean-field case are drastically changed by the short-range nature of the connectivity, so that no synchronization is expected in two dimensions [4–6]. An interesting analysis based on real space renormalization group theory [7] suggests an explicit expression for the lower critical dimension, below which one does not expect MME, the latter being the dynamical analogue of ferromagnetic order. A related situation occurs within the equilibrium statistical physics of spin systems with a continuous degree of freedom of the XY type, as already suggested in the past [4]. In the mean-field statistical mechanics of continuous spin models, the system has typically a spontaneous magnetization and the out-of-equilibrium behaviour is a very interesting problem [8]. These models closely relate to the Kuramoto model, where the additional presence of an intrinsic, quenched frequency distribution has to be taken into account. The system does not reach equilibrium but is self-driven in an out-of-equilibrium steady-state regime. Both the dynamical and equilibrium properties of continuous spin models change importantly when short-ranged, nearest-neighbour interactions are considered. For what concerns the equilibrium properties, rather than symmetry breaking, algebraic order (AO) occurs in the two-dimensional XY model. Namely, the system has infinite correlation length, but no magnetization at finite temperature [9–11]. When considering the case of low-dimensional arrays of coupled oscillators, the problem is further complicated by the fact that quenched distributed frequencies are present and the system is driven out of equilibrium, so that the standard tools of equilibrium statistical mechanics, apparently, cannot be exploited.

The analogy between arrays of identical oscillators and the statistical physics of XY type of models with continuous degrees of freedom has been already discussed and attempts to relate the dynamical properties of low-dimensional arrays of oscillators to the classical theory of dynamical critical phenomena [12, 13] have been made. In this work we show how the XY model in the presence of ad hoc disorders, relates to the problem of arrays of dissipatively coupled, short-ranged oscillators with a natural intrinsic frequency spread.

We will refer to this problem as the finite-dimensional Kuramoto model or lattice oscillator problem [4]. The simplest case one may consider is the strictly two-dimensional case, where the XY type of models have been studied within real space renormalization group theory [11] and with the field theoretical approach [14].

We will show that the correspondence between the dynamical behaviour of dissipatively coupled arrays of identical oscillators and model A of non-equilibrium critical phenomena [15] can be extended, in the two-dimensional case, to the problem of non-identical oscillators, so that the steady-state non-equilibrium properties of the system may be considered as a function of the quenched intrinsic frequency distribution. This will allow us to describe the steady-state, out-of-equilibrium properties of two-dimensional oscillator arrays, by means of classical

statistical mechanics. The possibility to consider non-identical oscillators is essential, in that a finite width of the intrinsic frequency distribution is the fundamental parameter one discusses in all Kuramoto type of models. We will also include a physical temperature, corresponding to Brownian noise in the array and consider the case of Gaussian and/or bimodal frequency distributions, options commonly considered in the context of the mean-field Kuramoto model [3].

The steady-state properties of two-dimensional arrays of dissipatively coupled non-identical oscillators are related, in the phase approximation, to the equilibrium properties of an effective XY model in the presence of random Dzyaloshinskii–Moriya (DM) interactions. The random DM interactions effectively induce phase canting between neighbouring oscillators, and can be related, as we will show below, to the quenched distribution of the native intrinsic frequencies. We will show how the steady-state, non-equilibrium properties of two-dimensional arrays of dissipatively coupled, non-identical oscillators (in the phase approximation) are related to the equilibrium properties of the two-dimensional XY model in the presence of disorder. A similar approach to describe the non-equilibrium properties of two-dimensional superconducting arrays with external currents by means of the equilibrium properties of the XY model in a field has been considered in the past [16].

Before we begin our analysis, it is important to stress that the properties of the Kuramoto model in low dimensions have already been discussed [4–7]. In particular one might argue at this point that in two dimensions no MME is expected, so that synchronization phenomena cannot occur. This argument reminds a similar one about the nature of the critical properties of the two-dimensional XY model [10, 11]. Even though no magnetization is expected at finite temperature, the two-dimensional XY model is known to present critical properties of a special type, namely AO. We will show in what follows that AO is expected in dissipatively coupled two-dimensional arrays of phase oscillators (and one might refer to it as algebraic synchronization). Dissipatively coupled oscillators in two dimensions are characterized by a phase transition of the Berezinskii–Kosterlitz–Thouless (BKT) type, as we will show below, even for non-identical oscillators, when one considers the case of distributed intrinsic frequencies. The main question we will address is how AO is affected by the disruptive effect of the intrinsic frequency distribution, and if a steady-state algebraic synchronization regime exists in two-dimensional arrays of oscillators.

We will answer these questions by evaluating the renormalization group recursion relations, within the position space Migdal–Kadanoff (MK) approximation, for the XY model with quenched random interactions of the DM type. We consider the standard formulation of the XY model within the MK approach [17], adopting the discretization scheme introduced in [19], and proposing a novel set of recursion relations that allow us to consider both exchange and DM type of interactions in a consistent way. The case of random quenched exchange and DM interactions will be considered according to the methods discussed in the context of the real space renormalization group of disordered spin systems [20]. We will consider a renormalization group rescaling length $b = 3$, to treat exchange and random DM interactions of random opposite signs on equal ground [21], generalizing the position space renormalization group (RG) recursion relations of the two-dimensional XY model to the case of both exchange and DM interactions. Together with the novel set of RG recursion relations, we present results obtained by means of an algorithmic method capable of implementing these recursion relations and evaluate the corresponding phase diagram, for different model systems, by fixing the proper choice of the initial conditions in the RG transformation. Choices one may consider within the proposed approach range from the XY spin glass (XYSG), where randomness is in the exchange interaction and no DM interactions are present, the two-dimensional ferromagnet with DM interactions (XYDM) problem, where randomness is in the DM term and no random

exchange interactions are present, and ultimately the random gauge glass (RGG) problem, as an important general case where both types of randomness are present.

The RG initial condition we discuss in relation to two-dimensional arrays of dissipatively coupled, non-identical oscillators, is, as we will show below, of the XYDM type. Clearly, randomness in the dissipative/exchange coupling term is also physically relevant, when discussing two-dimensional oscillator arrays, in that the strength of the dissipative coupling might vary across the system. Moreover we will show that any initial condition that includes randomness in the DM term will generate random exchange interactions under renormalization group transformation.

Our results do not limit the analysis to the oscillator array problem, but pertain to a variety of model systems that have been considered in the past and where the properties of the XY model in the presence of quenched random interactions have been discussed.

The phase diagram of the two-dimensional XY model in the presence of quenched random DM impurities is the subject of an interesting debate [23–26] we here address from a novel perspective and show to relate to the study of dissipatively coupled, two-dimensional oscillator arrays.

2. The model

The dynamical behaviour of oscillator arrays in the vicinity of a supercritical Hopf bifurcation can be formulated in terms of complex amplitude equations, describing the amplitude and phase of each oscillator. The complex amplitude is defined as $A_k = r_k \, e^{i\theta_k}$, where the index $k = 1, \ldots, N$ labels each of the $N = L \times L$ oscillators across the squared array. We consider the oscillators to be of the generalized Van der Pol form, so that, in the right-hand side of the complex amplitude equations, we include a damping term $\mu_k A_k$ as well as a nonlinear term $-\gamma_k A_k |A_k|^2$ that balances energy losses and induces self-sustained motion for each oscillator. We include stiffening nonlinearities of the Duffing type, quantified by the parameter α_k, and we assume each oscillator to have its own intrinsic frequency ω_k, which is a quenched, time-independent variable, distributed according to a given frequency distribution $P(\omega)$, centred around the typical frequency value ω_o (note that the specific value of ω_o is not important as one can absorb it via a global phase shift in the complex amplitude A_k). We include the presence of Brownian fluctuations across the array, so that an effective temperature T is introduced via a Langevin noise term $\eta_k(t)$, for each oscillator $k = 1, \ldots, N$, at time t, where $\langle \eta_k(t) \eta_{k'}(t') \rangle = 2T \delta_{k,k'} \delta(t - t')$. We restrict the range of the interactions to neighbouring oscillators on the square lattice, and $\mathcal{L}_k$ represents the set of four neighbouring sites to site k. We consider a reactive interaction R_{ij} as well as a dissipative interaction J_{ij} between neighbouring oscillators, and the following equations for the complex amplitude can be written [27, 28]:

$$\frac{dA_k}{dt} = (\mu_k + i\omega_k)A_k - (\gamma_k + i\alpha_k)|A_k|^2 A_k + \sum_{j \in \mathcal{L}_k} (J_{jk} + iR_{jk})(A_j - A_k) + \eta_k(t). \tag{1}$$

We begin our analysis considering the case of identical oscillators, and we set $\mu_k = \mu$, $\gamma_k = \gamma$ and $\alpha_k = \alpha$ for all oscillators $k = 1, \ldots, N$. We now rewrite the amplitude equations (1) that implicitly depend on the distribution of intrinsic frequencies $P(\omega)$. We recover the case of identical oscillators of frequency ω_o as soon as $P(\omega) = \delta(\omega_k - \omega_o)$. The typical distribution we consider in the case on non-identical oscillators is a Gaussian distribution $P(\omega) = \mathcal{N}(\omega_o, \Delta)$, with variance Δ around the average typical value ω_o. The coupling J_{ij} corresponds to dissipative coupling that we initially assume to be uniform, i.e. $J_{ij} = J$ for all pairs of neighbouring oscillators and vanishes otherwise. The case of reactively

coupled oscillators has been considered in [29], in the mean-field approximation, and, in that case, Duffing nonlinearities are shown to be important. Neglecting reactive terms and considering only coupling of the dissipative type, we rewrite the above amplitude equations (1) as

$$\frac{dr_k}{dt} = \left(\mu - \gamma r_k^2\right)r_k + J \sum_{j \in \mathcal{L}_k} r_j(1 - \cos(\theta_j - \theta_k)) + \eta_k^r(t)$$

$$\frac{d\theta_k}{dt} = \left(\omega_k - \alpha r_k^2\right) + J \sum_{j \in \mathcal{L}_k} r_j/r_k \sin(\theta_j - \theta_k) + \eta_k^\theta(t), \tag{2}$$

where η_k^r and η_k^θ are the Gaussian distributed Langevin noise terms. Including a reactive term means one should include $\sum_{j \in \mathcal{L}_k} R_{jk} r_k \sin(\theta_j - \theta_k)$ to the right-hand side of the first equation and $\sum_{j \in \mathcal{L}_k} R_{jk} r_j/r_k (\cos(\theta_j - \theta_k) - 1)$ to the right-hand side of the second. Together with reactive coupling terms, we will neglect, in what follows, the nonlinear terms of the Duffing type, related to the mechanical stiffening of the oscillators. A similar study has been considered, in the mean-field case, in [2], except we now consider explicitly the case of short-ranged interactions. The magnitude of the complex amplitude crosses over to a constant value, as soon as the values of μ and γ, related to the Van der Pol strength, are in the proper region of the parameter space. A similar discussion is well known in the context of the time-dependent Ginzburg–Landau equations, when discussing the dynamical behaviour of the XY model [31], and interestingly similar type of considerations have been discussed for the mean-field oscillator array problem [2]. Assuming the width of the intrinsic frequency distribution to be relatively high peaked around the average value ω_o, the above equations reduce to the phase approximation [2] and we write

$$\frac{d\theta_k}{dt} = \omega_k + J \sum_{j \in \mathcal{L}_k} \sin(\theta_j - \theta_k) + \eta_k^\theta(t). \tag{3}$$

In the case of identical oscillators, we recover the Langevin equations for the XY model. Differently, we are back to the two-dimensional oscillator lattice problem. As mentioned above, the dynamical behaviour of identical oscillators, reduces, up to a global redefinition of the complex amplitudes, to the coarsening dynamics of the XY model, so that the above equations (2) reduce to model A of critical dynamical phenomena [15, 30, 31] and the formal analogy between a critical Hopf bifurcation and phase transition theory has just been shown in this case. The dynamical behaviour of non-identical phase oscillators is certainly a challenging one. As usually happens when considering the presence of quenched random interactions in the context of statistical physics we want to understand how the pure model is affected by disorder. We are effectively studying the low-dimensional version of the Kuramoto model, and the first question we address is how the random frequency terms affect the onset of AO we know to be present in the case of identical phase oscillators.

The relevance of these models and of the above questions to the oscillator array problem can be explained as follows. Let us consider the following Hamiltonian:

$$-\beta\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \vec{S}_i \vec{S}_j + \sum_{\langle i,j \rangle} D_{ij} \hat{z} \cdot \vec{S}_i \times \vec{S}_j \tag{4}$$

Equation (4) can be written, for uniform exchange interactions $J_{ij} = J_o$, as [23]

$$-\beta\mathcal{H} = \sum_{\langle i,j \rangle} [J_o \cos(\theta_i - \theta_j) + D_{ij} \sin(\theta_i - \theta_j)]$$

$$= -\sum_{\langle i,j \rangle} \tilde{J}_{ij} \cos(\theta_i - \theta_j - q_{ij}), \tag{5}$$

where $\tilde{J}_{ij} = \left[J_o^2 + (D_{ij})^2 \right]^{1/2}$ and $q_{ij} = \arctan(D_{ij}/J_o)$. Following closely the considerations in [23], we consider the effective, average interaction

$$\tilde{J}_{ij}/k_B T \rightarrow J \equiv \left\langle \left[J_o^2 + (D_{ij})^2 \right] \right\rangle_d, \tag{6}$$

where $\langle \cdots \rangle_d$ indicates average over the randomness. In the continuous formulation, when $\theta_i \rightarrow \theta(\vec{r})$ and $q_{ij} \rightarrow \vec{q}(\vec{r})$, equation (4) reads, up to constant terms,

$$-\beta\mathcal{H} = -\frac{1}{2}J \int d^2r \, |\vec{\nabla}\theta(\vec{r}) - \vec{q}(\vec{r})|^2, \tag{7}$$

where $\vec{q}(\vec{r})$ is the random dipole term distributed according to the quenched distribution $P(\vec{q}(\vec{r}))$. It is then straightforward to derive the Langevin equation

$$\frac{d\theta(\vec{r})}{dt} = \omega_{\vec{r}} - J\nabla^2\theta(\vec{r}) + \eta_{\vec{r}}(t). \tag{8}$$

The correspondence between equations (8) and (3) is established observing that the quenched frequencies in the oscillator array problem are related to the random dipole field according to $\omega_{\vec{r}} \propto \vec{\nabla} \cdot \vec{q}(\vec{r})$, i.e. to the gradient of the quenched dipole field $\vec{q}(\vec{r})$ introduced above. Similar considerations on the Langevin equation for the XYDM model have been derived and discussed already in [39].

In the classical formulation of the two-dimensional XY model one expands around small values of subsequent phase shifts and then imposes the periodicity of the phase variables, leading to the canonical Coulomb gas formulation [44], so that the Kosterlitz–Thouless renormalization group equations can be derived. A similar analysis for the Coulomb gas in the presence of the random dipole moments $\vec{q}(\vec{r})$ has been derived [23]. Averaging over the disorder, e.g. with the replica method [23, 37], recursion relations of the Kosterlitz–Thouless type, which include the presence of disorder, have been derived and the phase diagram in the space of temperature and random strength has been shown to be reentrant [23].

Meanwhile we have just shown that the Langevin equations for the XYDM model are related to the oscillator lattice problem of equation (3). It is then natural to ask if the results obtained by renormalization group theory for the XYDM model apply to the oscillator lattice problem and if BKT order is expected in dissipative two-dimensional oscillator arrays. We will address these questions in the rest of the paper, showing how the non-equilibrium steady-state properties of the problem are indeed related to the equilibrium properties of an effective XY model with random quenched interactions.

In the following section, we will review results obtained by means of numerical simulations, i.e., by direct integration of equation (3). We will consider the problem from the perspective of real space renormalization group theory in section 4. In section 5, we will present the MK solution of the XY model with both exchange and DM interactions, deriving a novel set of recursion relations that generalize those already discussed in the past. Methods and techniques to implement numerically such recursion relations in the case of quenched random interactions are also discussed. In section 6, we discuss the relevance of the above set of recursion relations to the case of identical oscillators, while the study of non-identical oscillators can be found in section 7. Conclusions and future perspectives are confined to section 8.

3. Numerical simulations

In this section we discuss the results obtained by direct integration of equation (3). We used a fourth-order Runge–Kutta standard integration scheme, both in the case of identical oscillators, thus effectively recovering the properties of the classical XY model, and in the case
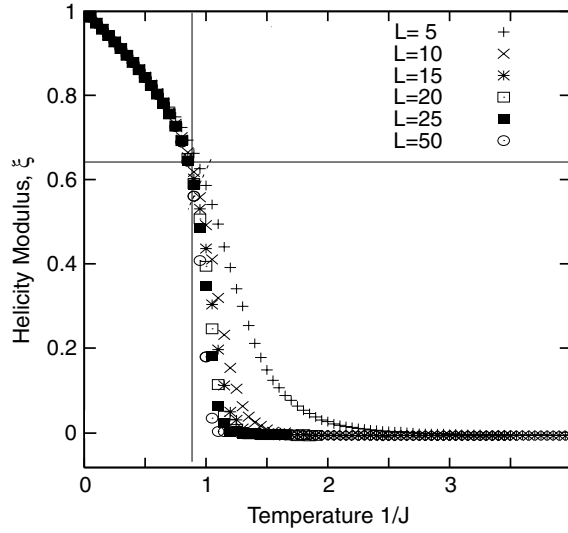
Figure 1. Helicity modulus computed for system sizes $L = 5, 10, 15, 20, 25, 50$ in the two-dimensional XY model without disorder, obtained via direct Runge–Kutta integration methods. The helicity modulus jump, occurring at the universal value $2/\pi$, indicated by the horizontal dotted line, becomes steeper for increasing system sizes. We could extrapolate a critical transition value of $T_{KT} \simeq 0.89$, indicated by the vertical solid line, in agreement with the results obtained in [35].

of non-identical oscillators, when quenched random intrinsic frequencies are present across the array. As mentioned above, we considered a Gaussian form of the intrinsic frequency distribution of variance Δ around a typical frequency value ω_o, explicitly solving the Langevin equation (3), via Runge–Kutta methods for a system of size $N = L \times L$, following the standard methods and re-deriving the results discussed in the literature [32, 33]. The same algorithm, in the case of mean-field interactions and when random frequency terms are included, reproduces the results of the Kuramoto model. We first discuss the results obtained in the case of identical oscillators. Even though the system does not order at finite temperature, phase correlation effects are present and are quantified by finite values of the helicity modulus [34], as predicted by the Kosterlitz–Thouless renormalization group theory. We report the results obtained for the helicity modulus in figure 1, while figure 2 shows the results obtained for the exponent η, which measures the algebraic decay of correlations in the system. Namely, we expect algebraic decay of the correlation function $\langle \theta_i - \theta_j \rangle \propto r^{-\eta}$, where r is the cartesian distance between sites i and j. The numerical results shown are consistent with the expected BKT transition [11, 35]. The algebraic decay of the correlation function can be measured and the weak violation of universality critical phenomena is seen, as expected, with the critical exponent being close to the expected value $\eta = 1/4$ at the transition point, as in figure 2.

We now discuss the results obtained when quenched random frequencies are present. We run the same algorithm discussed above for different quenched realizations of the frequency distribution, evaluating the helicity modulus in the system, and averaging over all realizations considered. We observed that for each given quenched realization of the intrinsic frequency distribution, the system reaches a steady state, i.e., for any realization of the disorder, we observed that the Runge–Kutta algorithm reaches a time-independent state. We computed the average helicity modulus over several realization of disorder, for a given system size ranging, as in the pure case, from $L = 5$ to $L = 50$, for two-dimensional arrays of size $L \times L$.
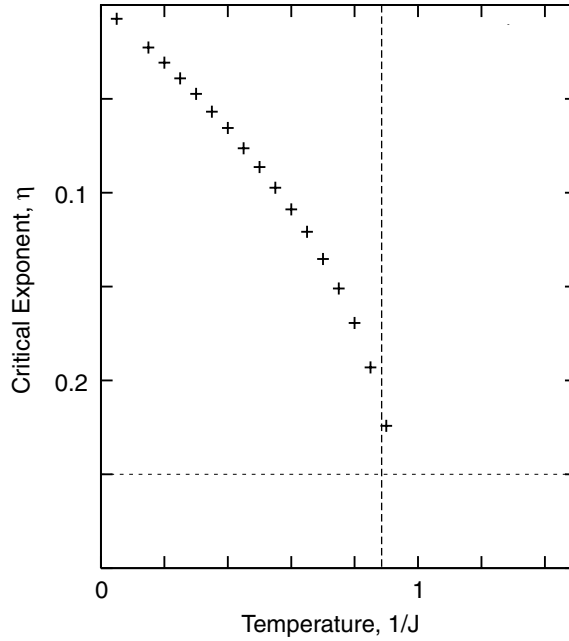
Figure 2. Critical exponent η. We computed explicitly the correlation function at increasing values of the temperature $T < T_{KT}$, observing, as expected, a power-law decaying correlation function. The estimated value of the exponent is consistent with the value $\eta = 1/4$, expected at $T = T_{KT}$ [11], and indicated by the dotted horizontal line. The vertical dotted line represents the location of the transition point, as obtained from figure 1.

We observed that the sample-to-sample fluctuations become very strong as the width of the intrinsic frequency distribution grows. For very small system sizes we observe the helicity modulus to be finite below the critical point, which decreases with the disorder strength, as in the inset of figure 3. For systems of size $N = L \times L = 32^2$ it becomes difficult to have sample-to-sample fluctuations under control, so no conclusive results were obtained for this system size and above, even after a run of several weeks on a standard workstation. Differently, in the mean-field case, where each oscillator is coupled to all others with equal strength, we were able to control sample-to-sample fluctuations and an average value of the Kuramoto order parameter was easily obtained, e.g. at zero temperature, consistently with theoretical predictions, up to relatively large system sizes $N = 10^4$. We note that, in this case, as discussed in the literature, to reproduce the Kuramoto order parameter as a function of the random width, the proper choice on the tail of the distribution $P(\omega)$ has to be made.

To conclude this section, concerning the direct integration approach to equation (3), we want to comment further the results we obtained in the presence of disorder. Our findings indicate that the location of the BKT transition decreases for increasing values of the disorder strength. On the other side, in the low-temperature region and for system sizes $L \geqslant 32$, we do not see the helicity modulus to reach a constant finite value as for smaller system sizes $L = 16$ analysed in figure 3. If AO is clearly seen to occur at finite temperatures, for finite, small values of the intrinsic frequency width distribution, numerics cannot state in a conclusive manner whether such transition extends all the way down to zero temperature for system of size $L \geqslant 32$. The helicity modulus sample-to-sample fluctuations are stronger in the low-temperature region, so unique conclusions on the form of the phase diagram, in the
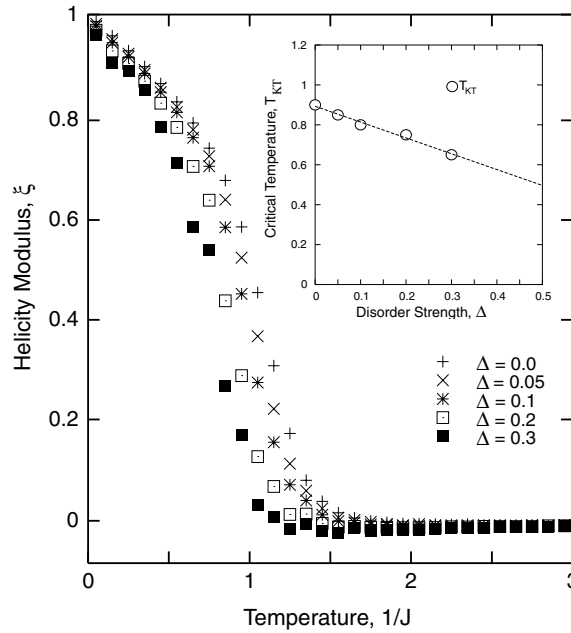
Figure 3. Helicity modulus computed for system size $N = 16 \times 16$. For this system size we were able to control sample-to-sample fluctuations and estimate the helicity modulus for increasing values of the disorder strength. Plus sign corresponds to $\Delta = 0$, crosses to $\Delta = 0.05$, stars to $\Delta = 0.1$, boxes to $\Delta = 0.2$ and filled boxes to $\Delta = 0.3$. The inset shows the location of the critical temperature T_{KT} together with the dashed line corresponding to the fit obtained, $T_{KT}(\Delta) \simeq T_{KT}(0) - 0.74\Delta$.

space of temperature and disorder strength, cannot be reached by means of direct integration methods. Interestingly, the sample-to-sample fluctuations weaken around, and clearly above, the transition point.

In this section, we have presented the numerical results obtained by direct integration methods of equation (3). In order to corroborate our findings we will now address the same questions just discussed, from the perspective of real space renormalization group theory.

4. Real space renormalization group theory

The effect of quenched random interactions on the Berezinskii–Kosterlitz–Thouless transition is an interesting problem that has been discussed in relation to a variety of low-dimensional physical systems. Infinitesimal bond randomness does not affect the transition, but it has not been clarified in a conclusive way how other types of quenched random interactions, namely site disorder and/or random interactions of the DM type, affect AO. In the case of random DM interactions, real space renormalization group considerations suggest that, at finite temperature, the BKT transition is not destroyed [23]. However the effect of quenched random DM impurities on the low-temperature behaviour of the system is usually expected to destroy BKT order, and a reentrant phase diagram has been derived, as discussed above, long ago [23, 36, 39]. On the other side, the validity of such results have been questioned in the recent past [25, 40], and the MK set of recursion relations of [19] has been reconsidered, possibly suggesting that AO is stable against disorder at finite, small temperatures so that no

reentrance would be present [25]. Before returning to this point, we want to stress why this question is also relevant when studying the effect of a random intrinsic frequency spread in the two-dimensional array of dissipatively coupled oscillators we consider in this work. We have shown above, when discussing the correspondence between equations (3) and (8), how the disorder in the intrinsic frequency distribution of the two-dimensional oscillator array problem is related to the DM type of quenched random interactions discussed in the context of the XY model. We infer that the non-equilibrium steady-state properties of two-dimensional arrays of dissipatively coupled array of oscillators can be understood studying the equilibrium properties of an effective XY model in the presence of quenched random interactions of the DM type. To do that, we will reconsider the MK position space renormalization group calculation of [25] to establish how random DM interactions affect AO, proposing a new set of recursion relations to evaluate the corresponding phase diagram. We suggest what we believe are the correct recursions that can answer the question of how quenched DM interactions affects BKT order, according to the MK approximation, a method that has been shown to describe low-dimensional systems with quenched disorder in a simple and effective way. We want to establish if the phase diagram of the XY model, in the presence of random DM interactions, obtained within the MK approximation, is consistent with the reentrant behaviour originally derived within real space renormalization group theory [23] and/if qualitative differences exist with these findings. A possibility we support is that reentrant behaviour occurs in the phase diagram but the phase boundary intercepts the zero temperature axis at finite values of the disorder strength, rather than at vanishing disorder strength as found in the analysis of [25]. In order to obtain the phase diagram, we write generalized recursion relations for a dicretized clock state model, following the work of [19], including bond and DM randomness in a consistent way, at any finite Q values of the clock state parameter. Our recursion relations reduce to the well-known random bond Ising model for values of the clock state parameter $Q = 2$ and relate to the MK recursion relations of the pure XY model at large values of the clock state parameter Q, when no disorder and no DM interactions are present. Without loss of continuity we are able to interpolate between the random bond Ising model (RBIM) [42], the pure XY model at large values of Q and no disorder as well as the general case of both exchange and DM random interactions being present, ultimately proposing a novel framework to study the random gauge glass model, where both exchange and DM interactions exist and are related by the proper initial condition in the renormalization group transformation. According to the discussion in the above section, the system of oscillators with distributed, quenched frequencies is related to the system of identical oscillators with an effective quenched dipole field, which induces a relative frequency mismatch. We now show how to write the correct recursion relations within the MK approximation, corresponding to the above Hamiltonian system (4). In the following section, we present a study of the two-dimensional XY model with quenched exchange and/or random DM interactions within the MK approximation. Our findings are relevant for the oscillator array problem and for many other physical problems to which the two-dimensional XY model in the presence of disorder, of the type dictated by equation (4), is known to relate.

5. The Migdal–Kadanoff method

In the classical formulation of the pure XY model with the MK renormalization group scheme [17], renormalization group recursion relations for the effective coupling of combined bonds, within a Fourier mode formulation, are considered [22]. At low temperatures the approximation recovers effective AO, the potential flows to a Villain form [18] and the system is characterized by an infinite correlation length, even though has vanishing magnetization.

The only drawback of the MK approximation is that after a sufficiently large number of renormalization group steps, the recursion relations flow to a high-temperature disordered sink at any finite temperature so that true fixed line behaviour is not present in the strict sense. The number of iterations required for this to occur is however diverging below the transition point so that effective AO is successfully described by the approximation. It is then reasonable to ask how to reformulate the classical MK approach to the XY model in the presence of quenched randomness.

5.1. The discrete clock state

Meanwhile, a second, interesting formulation of the MK approximation for the two-dimensional XY model has been suggested in the past [19]. Rather than the usual Fourier version of the MK recursion relations, we consider a discrete, clock state model, at integer finite values of the clock state parameter Q. The above formulation has been discussed already for the XY model in two dimensions. Attempts in order to include the presence of DM interactions have been made [45, 48, 40], but we are not aware of any study where correct recursion relations have been written for this case. Another delicate point is how one treats the recursion relations in the case of disorder (both for random exchange and DM interactions). We treat disorder consistently with the method introduced in the past for the RBIM, namely within a binning procedure. Clearly stochastic techniques [48, 40] are also an interesting option. The advantage of our approach is that the algorithm is a deterministic one, so we do not have to resort to pseudo-random numbers. The drawback is that the computational effort grows with the binning parameter involved and with the clock state parameter Q as we will discuss below. Another advantage is that we were able to check the results of the algorithm considering the case of the RBIM that has been studied already in the past [42]. The computational effort of the algorithm grows linearly with Q and with a cubic power in the number of bins, since we are considering a renormalization group rescaling length $b = 3$. As we will see for the simple case of the pure model, effective AO is observed already for values of the clock state parameter $Q \geqslant 16$. We now present a derivation of the renormalization group recursion relations.

5.2. The recursion relations

We consider the case of discrete phase angles $\theta_i = 2\pi q_i / Q$, $q_i = 0, \ldots, Q - 1$. We rewrite for convenience the Hamiltonian (4) that reads

$$-\beta\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \cos(\theta_i - \theta_j) + \sum_{\langle i,j \rangle} D_{ij} \sin(\theta_i - \theta_j). \tag{9}$$

We consider two generalized couplings between neighbouring sites $J(Q, q) = J_{ij} \cos(2\pi q / Q)$ and $D(Q, q) = D_{ij} \sin(2\pi q / Q)$ independently. The renormalization group recursion relations are simply obtained by explicitly performing a decimation in one spacial dimension according to the standard methods of real space renormalization group theory. We easily find

$$J'(Q, q) = \frac{1}{2} \log(R(Q, q)R^\dagger(Q, q)) - G'(Q)$$

$$D'(Q, q) = \log(R(Q, q)/R^\dagger(Q, q)) \tag{10}$$

$$G'(Q) = \frac{1}{2Q} \sum_{q=0}^{Q-1} \log(R(Q, q)R^\dagger(Q, q)),$$

where interaction terms are always considered modulo Q, where primed quantities refer to the renormalized interactions, and where the captive renormalization group constant $G(Q)$ imposes the constrain $\sum_q J'(Q, q) = 0$. The renormalization group polynomials, for the case of a rescaling length factor $b = 3$ read

$$R(Q, q) = \sum_{l,l'=0}^{Q-1} e^{\tilde{J}_{12}(Q,l)+\tilde{J}_{23}(Q,l')+\tilde{J}_{34}(Q,q+l+l')+\tilde{D}_{12}(Q,l)-\tilde{D}_{23}(Q,l')+\tilde{D}_{34}(Q,q+l+l')}$$

$$R^{\dagger}(Q, q) = \sum_{l,l'=0}^{Q-1} e^{\tilde{J}_{12}(Q,l)+\tilde{J}_{23}(Q,l')+\tilde{J}_{34}(Q,q+l+l')-\tilde{D}_{12}(Q,l)+\tilde{D}_{23}(Q,l')-\tilde{D}_{34}(Q,q+l+l')},$$

(11)

where the 'bond-moved' exchange interactions are

$$\tilde{J}_{ij} = \sum_{n=1}^{b} J_{i_n j_n}, \tag{12}$$

together with a similar expression for the DM interactions $\tilde{D}_{ij}$. These recursion relations might seem at first sight rather similar to those discussed in [19]; however, an important symmetry property that was not discussed in the above formulation has now been included, in that recursion relations are now explicitly written for each interaction term separately. A set of recursion relations of the MK type, where the two interactions were treated independently, was discussed in the so-called harmonic approximation [45–47].

The above recursion relations will be firstly discussed for a renormalization group rescaling length $b = 3$, in the absence of disorder, so we can check that the onset of effective AO described by the MK method is properly recovered within the discrete clock state formulation and we will also determine the effect of a uniform DM interaction on the pure XY model. We will then consider the case of small clock state Q values, and include the presence of disorder, observing that the above recursions reproduce, as expected, the phase diagram for the RBIM at $Q = 2, 4$, before returning to the issue of large values of Q and disorder being present.

Attempts in this direction were already considered in [19, 40, 48] except we are now able to incorporate the presence of DM interactions in a consistent way. The recursion relations (10), despite their simplicity, are the only possible ones that properly reflect the symmetries of the original Hamiltonian (4), as a direct analysis reveals. Note that, as also explicitly stated in [19], the symmetry properties of the generalized potential proposed there were lost under renormalization group transformation in the case of DM interactions being present, something that should simply not occur. The fact that the recursion relations written in the past were not complete is probably the reason behind the erratic behaviour of the renormalization group flows, also reported in [40], whenever interactions of the DM type were considered. Any conclusion on the shape of phase diagram of the $XYDM$ problem based on the recursion relations of [25] should then be taken with care.

Differently, in our method the potential maintains its symmetry properties under renormalization group transformation (see, e.g., figure 5), as it should be under any renormalization group (symmetry) transformation. We will discuss below how to implement the recursion relations (10) in the presence of quenched random interactions.

6. Identical oscillators

We first show the results we obtained for the problem in the absence of disorder, with a renormalization group rescaling length $b = 3$, at large, finite values of the clock state parameter
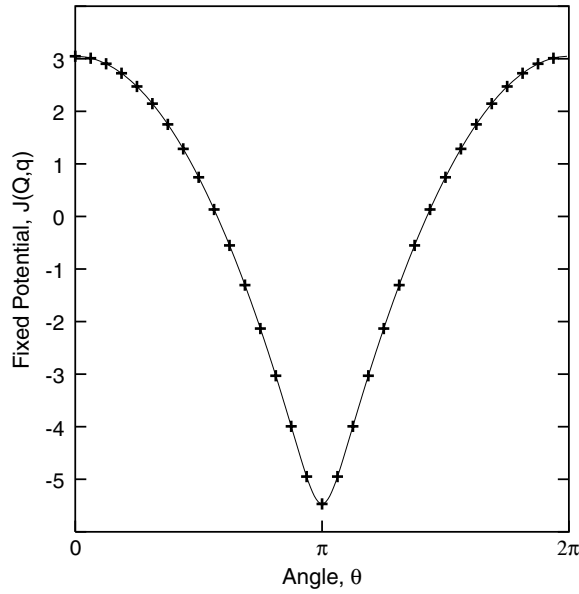
Figure 4. The fixed Villain potential $J(Q, q)$ observed for the pure XY model, at temperature value $T = 0.5$ for a renormalization group rescaling length $b = 3$, in the absence of DM interactions. Crosses show the fixed potential in the discrete scheme with $Q = 32$, while the solid line shows the results obtained for values of the clock state parameter $Q = 512$.

Q, checking that the results of the MK approximation in the XY model are properly reproduced within the discretization scheme considered here, discussing the role of DM uniform interaction terms, before returning on the role of disorder, which we will be able to include via the proper choice of the initial conditions of the renormalization group flows.

We report explicitly the form of the renormalization group polynomials for the case considered, corresponding to $b = 3$, which involves a double sum, as in equation (11), corresponding to the two-site decimation performed at each renormalization group step. When no randomness is present, we observe that the location of the Kosterlitz–Thouless transition, within the MK approximation, depends on the decimation parameter b, as can be seen in figure 6, showing the number of iterations needed before the above recursions (10) flow to the high-temperature disordered sink. Note that in the case of the pure XY model in the absence of uniform DM terms, we recover the usual Villain potential [18], at any value of the clock state parameter $Q \geqslant 16$. In figure 4 we show the quasi-fixed line potential $J(Q, q)$, where $\theta = 2\pi q/Q$, and $q = 0, \ldots, Q - 1$ we obtained for $Q = 32$ (crosses) and $Q = 512$ (solid line).

The above results show that effective AO is found at relatively small values of the clock state parameter, and we expect that the properties of the XY model are captured already by these relatively small values of Q. Already at values of $Q = 16$ we observe that in the low-temperature region the recursion relations (10) reach the quasi-fixed-line behaviour discussed for the XY model in the context of the MK approximation.

This last remark will be crucial when dealing with the disordered case. The computational effort of the algorithm we introduce in the following section scales in a linear way with Q, and scales as a cubic power of the number of bins we will use to discretize the quenched probability distribution(s) of the exchange and DM interactions at each renormalization group step.
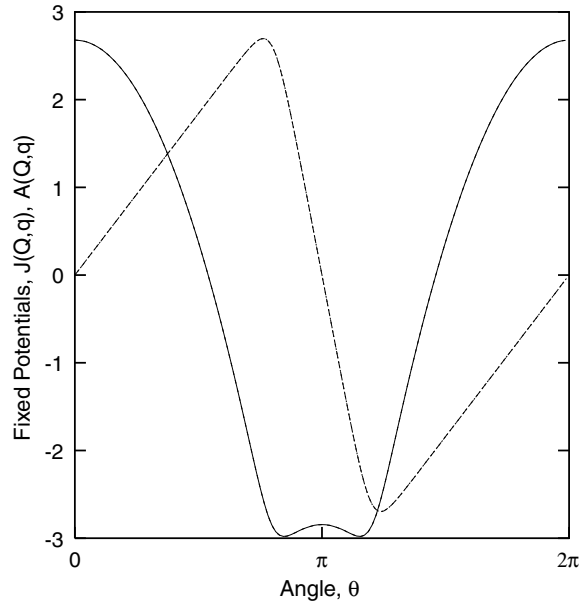
Figure 5. The two components of the complex fixed potential, observed for the pure XY model, at temperature value $T = 0.5$ for a renormalization group rescaling length $b = 3$, in the presence of uniform DM interactions, obtained for the clock state parameter value $Q = 512$. Clearly, under renormalization group transformation, the exchange interaction is symmetric around π, while the DM generalized interaction is antisymmetric. Fix-line behaviour as in the case where DM interactions are absent is observed.

In the presence of DM interactions the complex potential is defined as $[J(Q, q), A(Q, q)]$, where again $\theta = 2\pi q/Q$, $q = 0, \ldots Q-1$ and converges to a symmetric and an antisymmetric part, as dictated by the Hamiltonian of the problem (5). The symmetry properties of the two terms in the initial Hamiltonian are preserved by the above recursion relations (10), and that is because we wrote *two* distinct recursion relations for each term, since the renormalization group polynomials have to be regarded, in the presence of DM interactions, as complex quantities. As in the case of the pure XY model, effective AO is observed below the BKT transition point, which do not change in the presence of uniform DM interactions, as expected.

Before we conclude this section on the MK clock state approach to the pure XY model with uniform DM interactions, we want to add that, as stressed above, the location of the BKT transition does not changes importantly for values of the clock state parameter $Q > 32$, so we can safely consider the disordered case considering finite values of Q in this range of values.

For the case of the clock state parameter $Q = 4$ we observe a ferromagnetic transition point at $T_c \simeq 2.078$ that decreases for increasing values of the relative interaction strength D/J, as can be seen in figure 7.

7. The role of disorder

We begin considering the RBIM model that is obtained via the above recursion relations (10) at values of the clock state parameter $Q = 2, 4$, for vanishing values of the DM interaction
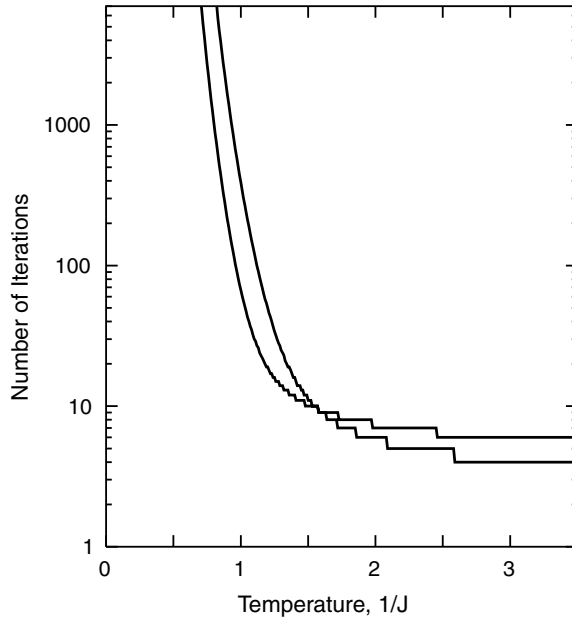
Figure 6. The number of iterations for the pure *XY* model with $b = 2$ and $b = 3$. The results show that effective AO is observed in that the iteration step number increase sharply around the effective critical transition point. The results show, as expected, that the location of the transition point depends on the renormalization group rescaling length.

strength *D*. The randomness in the exchange interaction is set by the initial condition in the renormalization group flows, chosen according to

$$P(J_{ij}) = p\delta(J_{ij} + J) + (1 - p)\delta(J_{ij} - J), \tag{13}$$

where the parameter *p* is the antiferromagnetic bond concentration. We now discuss the details of the binning procedure we used. The algorithm we propose introduces the probability density of exchange and DM interactions, via histograms, for each value of the clock state parameter $q = 0, \ldots, Q - 1$. To avoid the proliferation of the number of histograms we use a binning procedure, so after each pairwise operation, namely a bond moving or a decimation operation, the resulting *Q* distributions are evaluated for a given number of bins. Both exchange and DM interactions are evaluated with a binning procedure that independently treats positive and negative interactions, at each value of the clock state parameter. For interactions of each sign the grid size is determined by the standard deviation of the original distribution(s). Histograms that falls near a grid boundary are proportionally shared by neighbouring bins. The histograms falling outside the grid, that represent a small percentage of the overall probability, are treated independently in a single additional bin. For each value of the clock state parameter the binning procedure preserves the form of the probability distribution, and only the location of the histograms, not the relative probability, changes as a function of the clock state parameter value. Under renormalization group transformation, we explicitly check that the symmetry properties of the distribution(s) with respect to the clock state parameter are preserved. For both positive and negative interactions we construct a grid with $2B$ bins around the average value, so that the total number of bins is $2(2B + 1)$ for each interaction type. In the absence of DM interactions this implies that, having considered values of $B = 8$ and the renormalization group rescaling length $b = 3$, we have been dealing with the renormalization group flows
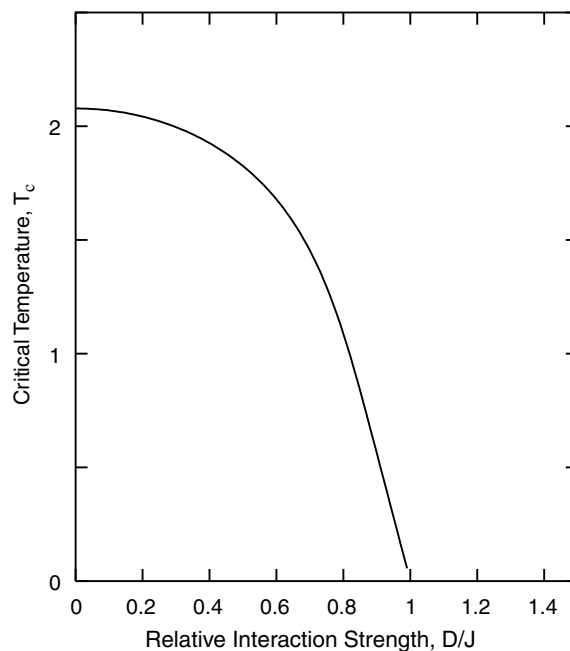
15

Figure 7. The location of the critical transition point for the clock state parameter value $Q = 4$, when no disorder is present, and where uniform DM interactions are considered. The x-axis shows the relative strength of the two interactions, as chosen in the initial conditions in the recursion relations (10).

of 39 304 histograms. In order to estimate the overall computational effort, this number has clearly to be multiplied by the clock state parameter Q, since the binning procedure is done independently for each value of $q = 0, \ldots, Q - 1$.

For values of the clock state parameter $Q = 4$, in the absence of DM interactions and for a bimodal distribution of the exchange interaction (13), we obtained the phase diagram in figure 8. The phase diagram is closely related, via a temperature redefinition, to the one discussed in the past [41, 42], corresponding to the $d = 2$ RBIM in the MK approximation. Note that an explicit discussion about the reentrant nature of the phase diagram in two dimensions can be found in [41]. The same, reentrant phase diagram however has already been obtained, as a direct inspection of figure 6 in [42] manifestly shows. The existence of reentrant phenomena, when discussing the phase diagram of low-dimensional random bond Ising systems, also observed in the three-dimensional case [42] within the MK approximation, has been confirmed recently by the transfer matrix approach [43]. The aim of this work is to confirm the presence of reentrant effects for spin systems with continuous XY symmetry, and their implications in the context of two-dimensional oscillator arrays. At $Q = 4$ and in the absence of DM interactions, gauge invariance holds, and we plot the Nishimori line [38]; note that the phase boundary reaches the highest value of the antiferromagnetic bond concentration on this line. It is important to mention at this point that any initial condition in the RG flows for $Q = 2, 4$, close to the boundary and above the Nishimori line, was observed to flow to the finite temperature unstable fixed point, while any initial condition below the Nishimori line was observed to flow to the strong coupling low-temperature fixed point. This phenomenon is known as strong violation of universality [42].
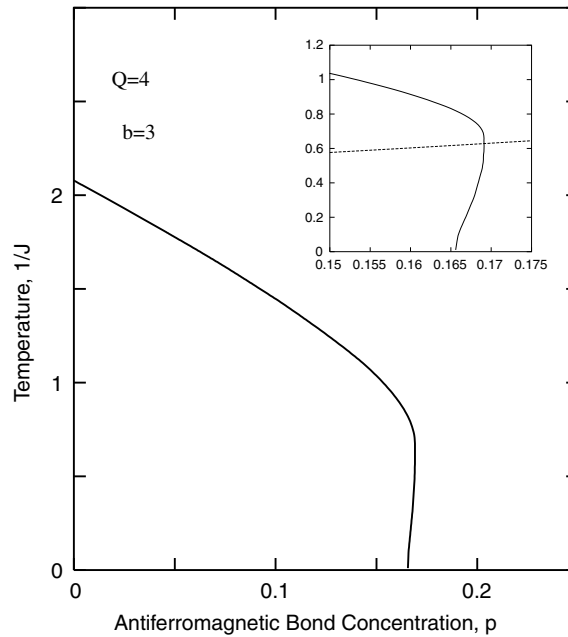
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Figure 8. The phase diagram for the clock state parameter value $Q = 4$, without DM interactions and in the presence of bimodal exchange interactions, with an antiferromagnetic bond concentration p. The inset shows the reentrant behaviour of the phase boundary around the multicritical point. The dotted line in the inset corresponds to the Nishimori symmetry line, $J = Q/4 \log((1-p)/p)$.

It is interesting to study at this point the above model at small, finite values of the parameter $Q > 4$ with disorder, since it is a well-defined problem that interpolates between the Ising and XY type of models. As found in the pure case, when disorder is present in the exchange interaction, we observe ferromagnetic order at low temperatures for the values $Q = 2, 4, 8$, while effective AO appears at values of the clock state parameter $Q \geqslant 16$. This means that the recursion relations (10) flow to a ferromagnetic fixed point for values of $Q \leqslant 8$, while at $Q \geqslant 16$ we observe the usual quasi-fixed line behaviour discussed in the context of the XY model, as shown in figure 10, and we do not expect things to change importantly for higher values of the clock state parameter Q, as we have seen already for the pure case, meaning that the discretization scheme converges rapidly to the model with continuous degrees of freedom. For the case of $Q = 8$, and when DM interactions are not present we computed, for a bimodal exchange interaction (13), the phase diagram in figure 9. In this case gauge invariance is not present [38] and reentrance does not occur, as expected. On the other side we still observe ferromagnetic and paramagnetic order, divided by the phase boundary (continuous thick line in figure 9), where a multicritical point occurs. As in the $Q = 4$ case discussed above, any initial condition close to the phase boundary and above the multicritical point flows to an unstable finite temperature fix point, while any initial condition close to the boundary and below the multicritical point flows to a strong coupling zero temperature distribution, even though gauge symmetry is not present. This implies that at small, finite values of the clock state parameter the strong violation of universality, discussed in the past for the RBIM [42], also occurs in the XY clock state model, as we saw for values of the clock state parameter $Q \leqslant 8$.
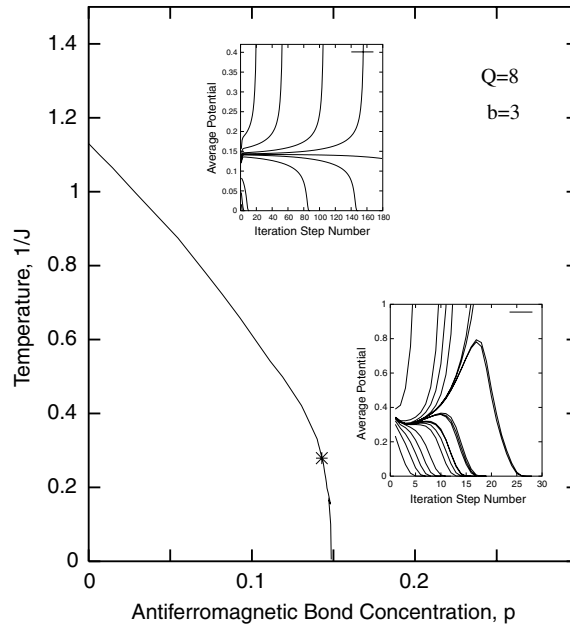
Figure 9. The phase diagram for the clock state parameter value $Q = 8$, without DM interactions and in the presence of bimodal exchange interactions (13), with an antiferromagnetic bond concentration p. Gauge invariance is not present in this case, since DM interactions are set to zero. Reentrance is not expected and the Nishimori symmetry line is not present. We observe the presence of a multicritical point. Any initial condition above the multicritical point and close to the phase boundary flows to a finite temperature unstable fix point, as seen in the inset above, while any initial condition below the multicritical point (indicated by the star symbol) and close to the boundary flows to a strong coupling, low-temperature fix distribution, as in the second inset. The insets show as a function of the RG iteration number, the average potential, defined as $\sqrt{\sum_q |J(Q, q)|^2}$.

In the case of larger values of the clock state parameter, e.g. $Q = 16$, and still at $D = 0$, we do not observe the recursion relations (10) to flow to a ferromagnetic fix point anymore, but rather quasi-fixed line behaviour of the BKT type appears at finite values of the random bond concentration, as seen in figure 10.

We are considering in fact the discrete version of the XY spin glass problem. No gauge invariance nor reentrance is expected in this case, and we observe effective AO at low values of the bimodal exchange interaction, while a paramagnetic phase occurs at higher values of disorder and temperature. No spin-glass order is expected in two dimensions anywhere in the phase diagram.

We did not consider yet the final case where both randomness in the exchange and DM interactions are present, simply because of the computational effort required. Stochastic methods would be a possibility to consider in this respect.

We believe we have set up a consistent machinery to evaluate the phase diagram in the general case of large values of Q and disorder of different types and the recursion relations we introduced, are the correct recursion relations to study the presence of reentrant phenomena within the MK approximation. The issue that remains to be considered is an algorithmic implementation that is characterized by a lower computational effort than the one here proposed. In our approach, when both disorder in the exchange and DM interactions is present, the renormalization group recursions involve a two-dimensional probability distribution and
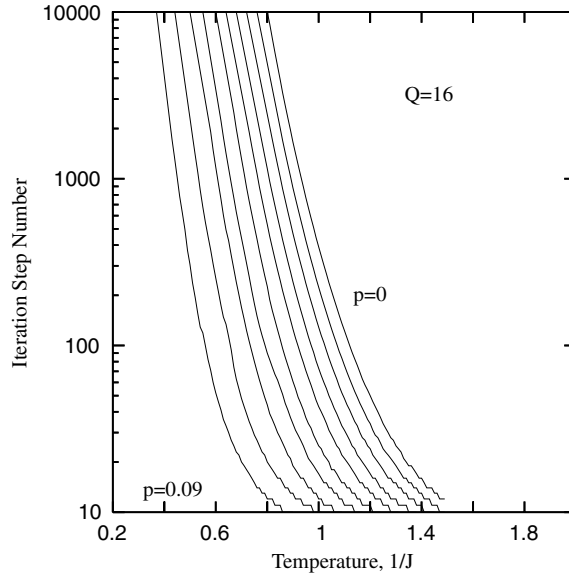
Figure 10. Number of iterations required to flow to the paramagnetic sink, as a function of temperature for the clock state parameter value $Q = 16$, for different values of the antiferromagnetic bond concentration $p = 0.0$, $p = 0.01, \ldots, 0.09$. The figure indicates that effective AO is observed for finite, small values of the disorder strength in the exchange interaction. From the location of these lines it is possible to reconstruct the phase diagram of the $XYSG$ problem at $Q = 16$.

the computational effort scales as $(4B + 2)^6$, so even a small value of the grid size $B = 3$ implies that the flows of up to 7529 536 histograms have to be considered, for each value of the clock state parameter.

8. Conclusions and future perspectives

We conclude with a few considerations on what we expect to see in the general case of large values of the clock state parameter Q and disorder of the DM and exchange type, according to the results obtained in this work.

If both randomness in the exchange and DM interactions are considered in a way such that gauge invariance is recovered, we expect to observe a BKT phase and reentrant behaviour of the phase boundary of the type discussed above. We expect the phase boundary to reenter below the Nishimori line, and finally to intercept the zero temperature axis at finite, non-vanishing values of the disorder strength. This would imply that AO is stable against the intrinsic frequency distribution and it is then implied that AO of the BKT can be observed in arrays of two-dimensional, non-identical oscillators, where technically one should consider temperature values, corresponding to Brownian fluctuations across the array, to be small. The reason why previous calculations within the MK approximation have not been able to show the reentrance we expect is simply because the recursion relations considered in [19, 40] were not taking properly into account the DM interactions. This is likely the reason why the MK phase diagram discussed in [25] is not reentrant, as we also observe for the case of $Q = 8$ in figure (9) where the DM interaction is explicitly set to zero and gauge invariance is not present. A careful analysis of the recursion relations (10) in the presence of DM interactions, and the
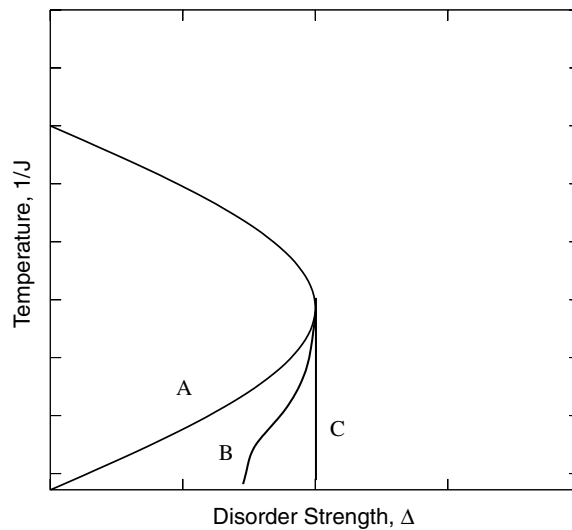
19

Figure 11. Qualitative behaviour of the phase diagram in the case of the two-dimensional *XY* model with DM random interactions of width Δ. (A) Real space renormalization [23], suggesting reentrance that extends all the way down to vanishing disorder strength. (B) Reentrant behaviour of the phase boundary and a BKT phase that extends at finite values of the disorder strength for low-temperature values.(C) The absence of reentrance and vertical boundary at finite values of the disorder strength as suggested in [25].

corresponding phase diagram, both at small and large values of the clock state parameter Q would be an interesting step forward to answer these questions in a quantitative manner.

The results we expect can be summarized as follows. The phase diagram is reentrant as derived long ago, but reentrance (see curve B in figure 11) reduces and the phase boundary between the BKT phase and the paramagnetic phase intercepts the zero temperature axis at finite values of the disorder strength. This intermediate scenario is consistent both with the reentrant behaviour derived within classical renormalization group arguments [23] and with the possibility that the phase boundary intercepts the zero temperature axis at finite values of the disorder strength.

Acknowledgments

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